Defaultable Bonds under Imprecise Information

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Abstract

This paper develops a structural model for defaultable bonds in a fuzzy environment. The numerical results calculated from the closed-form solution show that the fuzziness of the stochastic underlying asset and of bankruptcy costs have material impact on the term structures of credit spreads and on the duration of defaultable bonds.

1 Introduction

The valuation of defaultable bonds is at the center of financial research. Starting from Merton (1974), models of default risk have developed into a huge literature that has evolved around two main streams: the structural and the reduced-form models. In this paper we follow the structural approach whose advantage is to model the mechanism behind default events explicitly. Our aim is to show that the introduction of fuzziness helps to overcome some of the difficulties of the structural approach in reproducing empirical observed features. One of the major drawbacks of structural modeling is the unavailability of reliable accounting data. In practice, "bond investors cannot observe the issuer’s assets directly and receive instead only imperfect accounting reports" (Duffie and Lando, 2001). In this model the state variable that represents the assets of the firm is modeled as a fuzzy stochastic process. The reason behind this assumption is related to incomplete information. Although investors do not observe the true value of the assets of the firm, and thus the true indebtedness ratio, they have subjective beliefs about the reliability of the accounting data of the firm. Thus, fuzzy modeling adds a source of uncertainty to the classical stochastic modeling of the fundamental variable driving default and one interpretation is in terms of incomplete information. Another source of uncertainty incorporated in this model is the fractional recovery of asset value upon default that is modeled as a fuzzy number. The width of the fuzzy set is obtained from the empirical literature on bankruptcy (see Altman and Hotchkiss, 2006). Since we want to point out the advantage of a fuzzy approach, we adopt a parsimonious model which is a slight generalization of Merton (1974), in that bankruptcy costs are incorporated. The powerful contribution of fuzzy modeling comes explicitly to the light, as the technicalities of the recent literature on defaultable bonds are deliberately avoided. For example, some outcomes on bond duration that are precluded by classical structural modeling can be obtained as a result of fuzzy assumptions, as Section 4 shows. Another interesting outcome is the numerical computation of the average credit spreads for different credit ratings and the determination of their term structures. Eom et al (2004) have emphasized that
structural models always under-predict the spread on short-term and high credit quality bonds. Although the adoption of a structural model shares the difficulty of the other structural models in reproducing the empirical observed credit spreads, the introduction of fuzziness helps to improve the performance of the model since it widens the range of the possible outcomes. Therefore this paper, while stressing the flexibility of a fuzzy framework, provides new insight for future research on defaultable bonds taking into account the recent issues of credit risk analysis.

The remaining of the paper is organized as follows. The main formula is presented in Section 2 and is implemented in Section 3. Section 4 analyzes the problem of the duration of defaultable bonds and Section 5 computes fuzzy goals. Finally Section 6 concludes.

2 The fuzzy stochastic valuation model

In what follows let us introduce the basic fuzzy-stochastic elements that are useful for our application (see also Zadeh, 1965; Dubois and Prade, 1980; 2000). A fuzzy number is a fuzzy set (depicted with tilde) of the real line $R$, which is commonly defined by a normal, upper-semicontinuous, fuzzy convex membership function $\mu : R \rightarrow [0, 1]$ of compact support. The $\gamma$-cut of a fuzzy number is given by $\tilde{\mu}_\gamma = \{ x \in R \mid \mu(x) \geq \gamma \}$, $\gamma \in (0, 1)$ and $\tilde{\mu}_0 = cl \{ x \in R \mid \mu(x) \geq 0 \}$, where $cl$ denotes the closure of an interval. Let us write the closed intervals as $\tilde{\mu}_\gamma = [\tilde{\mu}_\gamma^-, \tilde{\mu}_\gamma^+]$ for $\gamma \in (0, 1)$. Given two fuzzy numbers, $\tilde{\mu}$ and $\tilde{\eta}$, the partial order $\succeq$ on fuzzy numbers can be defined such that $\tilde{\mu} \succeq \tilde{\eta}$ means that $\tilde{\mu}_\gamma \geq \tilde{\eta}_\gamma$ and $\tilde{\mu}_\gamma^- \geq \tilde{\eta}_\gamma^+$ for all $\gamma \in (0, 1)$. The arithmetic operations on two fuzzy numbers can be defined in the standard way, in terms of the $\gamma$-cuts for $\gamma \in (0, 1)$. In particular, for fuzzy numbers $\tilde{\mu}$ and $\tilde{\eta}$ the addition and subtraction $\tilde{\mu} \pm \tilde{\eta}$ and the scalar multiplication $a\tilde{\mu}$, where $a \geq 0$, are fuzzy numbers as follows:

$$(\tilde{\mu} + \tilde{\eta})_\gamma = [\tilde{\mu}_\gamma^- + \tilde{\eta}_\gamma^-, \tilde{\mu}_\gamma^+ + \tilde{\eta}_\gamma^+],$$
$$(\tilde{\mu} - \tilde{\eta})_\gamma = [\tilde{\mu}_\gamma^- - \tilde{\eta}_\gamma^+, \tilde{\mu}_\gamma^+ - \tilde{\eta}_\gamma^-],$$
$$(a\tilde{\mu})_\gamma = [a\tilde{\mu}_\gamma^-, a\tilde{\mu}_\gamma^+].$$

Moreover, multiplication between two fuzzy numbers $\tilde{\mu}$ and $\tilde{\eta}$ is given by:

$$(\tilde{\mu}\tilde{\eta})_\gamma = \left[ (\tilde{\mu}\tilde{\eta})_\gamma^- , (\tilde{\mu}\tilde{\eta})_\gamma^+ \right],$$
where $(\tilde{\mu}\tilde{\eta})_\gamma^- = \min \left[ \tilde{\mu}_\gamma^- \tilde{\eta}_\gamma^- , \tilde{\mu}_\gamma^+ \tilde{\eta}_\gamma^- , \tilde{\mu}_\gamma^- \tilde{\eta}_\gamma^+ , \tilde{\mu}_\gamma^+ \tilde{\eta}_\gamma^+ \right]$ and $(\tilde{\mu}\tilde{\eta})_\gamma^+ = \max \left[ \tilde{\mu}_\gamma^- \tilde{\eta}_\gamma^+ , \tilde{\mu}_\gamma^- \tilde{\eta}_\gamma^- , \tilde{\mu}_\gamma^+ \tilde{\eta}_\gamma^- , \tilde{\mu}_\gamma^+ \tilde{\eta}_\gamma^+ \right].$

A fuzzy-number-valued map $\tilde{X}$ is called a fuzzy random variable if

$\{ (\omega, x) \in \Omega \times R \mid \tilde{X}(\omega)(x) \geq \gamma \}$

for all $\gamma \in (0, 1)$. It is called integrably bounded if both $\omega \rightarrow \tilde{X}_\gamma^-(\omega)$ and $\omega \rightarrow \tilde{X}_\gamma^+(\omega)$ are integrable for all $\gamma \in (0, 1)$. The expectation $E(\tilde{X})$ of the integrably bounded fuzzy random variable $\tilde{X}$ is also defined by a fuzzy number.
$E(\tilde{X})(x) = \sup_{\gamma \in (0,1)} \min \left\{ \gamma, I_{E(\tilde{X})}(x) \right\}, \ x \in R,$

where $E(\tilde{X}) = \int_\Omega \tilde{X}_\gamma^-(\omega)dP(\omega) = \int_\Omega \tilde{X}_\gamma^+(\omega)dP(\omega)$, $\gamma \in (0,1)$.

Let us now introduce the valuation method based on fuzzy variables. The framework is the structural or value-of-the-firm approach, where the fundamental variable is the value $V$ of the (unlevered) firm’s assets:

$$dV_t = V_t(\nu dt + \sigma dW_t)$$

where $\nu$ is the appreciation rate, $\sigma$ is the volatility ($\nu \in R, \sigma > 0$) and $W_t$ is a standard Wiener process. Let $\{V_t\}_{t \geq 0}$ be a fuzzy stochastic process, which is defined as follows:

$$\tilde{V}_t(\omega)(x) = 1 + \frac{x-V_t(\omega)}{\alpha^-_t(\omega)}, \text{ if } V_t(\omega) - \alpha^-_t(\omega) \leq x \leq V_t(\omega)$$

$$1 - \frac{x-V_t(\omega)}{\alpha'^+_t(\omega)}, \text{ if } V_t(\omega) \leq x \leq V_t(\omega) + \alpha'^+_t(\omega)$$

$$0, \text{ otherwise}$$

with $\alpha^-_t(\omega) \geq \alpha'^+_t(\omega)$. That is, the fuzzy random variable $\tilde{V}_t$ is of the triangular type, with centre $V_t(\omega)$, left-width $\alpha^-_t(\omega)$ and right-width $\alpha'^+_t(\omega)$. The assumption $\alpha^-_t(\omega) \geq \alpha'^+_t(\omega)$ is related to the investors’ subjective belief about the reliability of the accounting data of the firm: they tend to shift the known firm value to the left, because the published value of $V$ is more likely to be overstated than understated as mis-reporting or fraudulent behavior are a possibility. Observe that the fuzziness in the process increases as $\alpha'^+_t(\omega)$ become bigger. The $\gamma$-cuts of $\tilde{V}_t(\omega)(x)$ are

$$\tilde{V}_{t,\gamma}(\omega) = \left[ \tilde{V}_{t,\gamma}^-(\omega), \tilde{V}_{t,\gamma}^+(\omega) \right] =$$

$$= \left[ V_t(\omega) - (1-\gamma)\alpha^-_t(\omega), V_t(\omega) + (1-\gamma)\alpha'^+_t(\omega) \right].$$

We assume that the firm has only common stock and a zero-coupon bond outstanding. Let $B$ be the face value of the debt and $T$ the maturity date. We assume that the curve of bond prices is exogenously given and that the short-term interest rate $r$ is given. To avoid complications that would divert the analysis from its main aim - the impact of fuzziness- we adopt the default explanation postulated by Merton (1974): a firm defaults at debt maturity if assets are not sufficient to pay off the debt. Throughout the model the absolute priority rule is assumed, that is, shareholders get nothing in case of default. Bondholders are the priority claim holders but, upon bankruptcy, default costs may result in bondholders receiving only a fraction of the total firm value. We denote by $\delta$ the fractional recovery rate of firm value at default. This assumption results in a more general model than Merton (1974), where $\delta = 1$. In what follows, we assume that the recovery rate is a fuzzy number. Specifically,
\[ \tilde{\delta}(x) = \max \{1 - |x - \delta|/\beta\}, \]

that is, the fuzzy number \( \delta \) has a symmetric triangle-type shape, with centre \( \delta \) and width \( \beta \geq 0 \). The rationale behind a fuzzy recovery rate lies in the difficulty of getting a precise estimate of the actual recovery rate of defaultable bonds. Empirical investigation have been conducted to estimate bankruptcy costs (see Altman and Hotchkiss, 2006, Bris et al., 2006) and the actual recovery rate of defaulted bonds. Altman and Hotchkiss (2006) report several estimates of direct and indirect costs of financial distress: the latter are not directly observable and a methodology for measuring them was provided in Altman (1984) for the first time, where overall distressed costs are estimated around 15% of firm value. Thus, in our numerical computation, \( \delta \) is set at 85%. Some recent estimates of loss rates to investors upon default are reported in Altman and Hotchkiss (2006). By modelling the recovery rate as a fuzzy number one can take into account the financial operators’ subjective judgement.

Under standard assumptions (no arbitrage opportunities, perfect asset divisibility, continuous trading, no transaction costs, no taxes or informational asymmetries), it is straightforward to derive a pricing formula for the bond whose contractual features have been described above. Let \( D(V, \delta) \) denote the value of this bond at time 0, contingent on the value of \( V \) and \( \delta \). We evaluate the fuzzy stochastic process of the bond by means of the expectation \( E(.) \) with respect to the equivalent martingale measure:

\[
\tilde{D}_T(V, \delta) = E(e^{-rT}[B\tilde{V}_{T,\delta}^+ I_{\tilde{V}_{T,\delta}^+ \geq B} + (\delta\tilde{V}_T)^+ I_{\tilde{V}_T < B}] \mid V_0 = V; \delta)
\]

where \( I_X \) denotes the characteristic function of the set \( X \).

Following Yoshida (2002, 2003) we introduce a reasonable assumption A.1, which allows us to simplify the formulas although it is not necessary for the argument.

**Assumption A.1.** The stochastic process \( \alpha^\pm_t(\omega) \) is specified by \( \alpha^\pm_t(\omega) = c^\pm V_t(\omega) \), where \( 0 < c^+ \leq c^- < 1 \). The value \( \beta \) is specified by \( \beta = b\delta \), where \( 0 < b < 1 \).

Assumption A.1. is reasonable since \( \alpha^\pm_t(\omega) \) is related to the fuzziness of the volatility \( \sigma \) and the firm value \( V_t(\omega) \): actually, we represent by \( c^\pm \) the fuzziness of the volatility \( \sigma \), so that \( \tilde{V}_{t,\gamma}^\pm(\omega) = (1 \pm (1 - \gamma)c^\pm) V_t(\omega) \).

Similarly, \( \tilde{\delta}_{t,\gamma}^\pm = (1 \pm (1 - \gamma)b)\delta \).

We can prove Proposition 1:

**Proposition 1** Suppose that Assumption A.1 holds. Let \( \gamma \in (0, 1) \). The rational fuzzy price of a defaultable bond is given by:

\[
\tilde{D}_T(V, \delta) = B e^{-rT} N(d_2^\pm) + (1 - N(d_2^\pm)) [V\delta(1 \pm (1 - \gamma)(c^\pm + b \pm c^\pm b(1 - \gamma)))],
\]
where \( N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-w^2/2} dw \) is the standard normal distribution function and \( d_1^\pm \) and \( d_2^\pm \) are given by

\[
d_1^\pm = \frac{\ln(1(1-\gamma)c^\pm) + \ln(V/B) + T(r + \sigma^2/2)}{\sigma \sqrt{T}} + \sigma \sqrt{T},
\]

\[
d_2^\pm = d_1^\pm - \sigma \sqrt{T}.
\]

**Proof.** By a standard argument (see Merton (1974), equation (83)) one can see that \( \tilde{D}_\gamma^\pm(V, \delta) \) satisfy the following partial differential equation:

\[
\partial_t \tilde{D}_\gamma^\pm + \frac{1}{2} \sigma^2 V^2 \partial^2_\gamma \tilde{D}_\gamma^\pm + rV \partial_\gamma \tilde{D}_\gamma^\pm = r \tilde{D}_\gamma^\pm
\]

with the final condition \( B \) if \( \tilde{V}_\gamma^\pm \geq B \) and \( (\delta \tilde{V}_\gamma^\pm) \) in the opposite case. Then our formula is obtained as the solution of this Cauchy problem, if we take into account the expression for \( \gamma \)-cuts of fuzzy numbers.

3 Numerical implementation of the model

In this section the model is implemented for calculating the bond prices and credit spreads. Computation and plots can be easily obtained employing *Mathematica* that has built-in-routines allowing us to work out our closed-form solution. As the base case, we take a triangle-type shape function for the fuzzy firm value and assume an asymmetry that accounts for the investors’ subjective beliefs about the reliability of the accounting data of the firm. Since mis-reporting would more likely overstate the true asset value, then the investors’ valuation shifts the publicly known value of \( V \) to the left (see Figure 1). Also the fractional recovery rate of firm value at default, \( \delta \), is assumed to have a symmetric triangle-type shape, whose centre at 85% is prescribed by the empirical investigation about the bankruptcy costs, \( 1 - \delta \) (see Figure 2). Figure 3 depicts the resulting function for the fuzzy bond price in a base case with \( V = 300 \), \( B = 100 \) and maturity three years.

Now the model is calibrated to be consistent with historical data of different rating categories. The analysis is restricted to A- and B-rated firms to study how the model works on speculative and investment grade bonds. The values of the quasi-leverage ratio and the volatility are taken from Huang and Huang (2003). Our aim is to investigate the impact of a fuzzy framework on the term structure of credit spreads. The \( T \)-maturity of credit spread is defined as usual by \( s_T(V, \delta) = -\frac{1}{T} \ln(D(V, \delta)) - r \).

In Figures 4-7 the possible fuzzy shapes of the term structure of credit spreads are plotted for the two credit qualities considered above. The risk-free rate is fixed at 4%. The first two graphs (Figures 4 and 5) depict the membership function for the credit spread as the time to maturity varies. Figures 6 and 7 plot the term structure of the credit spread: the dashed curves delimit the fuzzy spread, the thin solid curve is the
Figure 1:

Figure 2:

Figure 3:
average of the extreme fuzzy values and the thick solid curve gives the crisp value. The typical shape of the term structures are initially upward sloping, have a hump and then decline. A striking thing is that fuzzy modeling predicts a wide-spread range of values for short maturities. In other words, while this model shares the unsatisfactoriness of structural modeling in calculating realistic short terms spreads, on the other hand, it points out this defect explicitly and at the same time provides a range of values for the short-term default spread. Figures 8 and 9 display the "fuzziness" of the credit spreads in terms of time to maturity, where the maximum width of the fuzzy set is taken as a proxy for the "fuzziness". Note that the "fuzziness" increases with the shortness of the maturity, especially in the case of the speculative grade. Therefore, the commonly acknowledged shortcoming of structural models - to be able to predict sufficiently large spreads on short term- seems to be mitigated in a fuzzy framework.

4 The duration of a fuzzy bond

An important issue in the theory of corporate bonds is the investigation of the sensitivity of a bond price to interest rates and the comparison between the duration of a defaultable bond and an equivalent default-free one. Two bonds are considered equivalent if they promise the same payments at the same dates. The first result about this subject is Chance (1990), showing that defaultable zero-coupon bonds are less sensitive to interest rates than their default-free counterparts. This result is extended in Agliardi (2007) to coupon-bearing bonds in a struc-
Figure 5:

Puzzly credit spread for B-rated bonds ($\tau = 4\tau$)

Figure 6:

Credit spread (bps) [A-rated]

Figure 7:

Credit spread (bps) [B-rated]
Figure 8: 

Figure 9: 

9
tural setting. However, empirical investigation points out the possibility of opposite outcomes. Jacoby (2003) reports that the duration of a non-callable corporate bond may be longer than its Macaulay counterpart. Thus, quoting Kraft and Munk (2007), "the statements on the duration of a corporate bond seem to be conflicting". In Kraft and Munk (2007) the reduced-form framework of Duffie and Singleton (1999) is adopted and it is shown that the duration of a defaultable coupon bond is generally smaller than that of the equivalent default-free bond, but the opposite is possible if the default intensity is sufficiently correlated with the default-free interest rate. This section adopts a different framework and shows that the duration puzzle is compatible with fuzzy modelling. More importantly, it is the fuzziness of bond valuation that accounts for both conflicting outcomes, that is, the smaller sensitivity of defaultable bonds to interest rates, which is forecasted by the structural literature, and the possible larger sensitivity that appears in some cases. Adopting our fuzzy model and the standard price-elasticity definition of duration with respect to the short rate, that is \( \text{dur} = \frac{1}{D} \frac{\partial D}{\partial r} \), we find that

\[
\text{dur} = (T B e^{-rT} N(d_2) - (1 - \delta) V \sigma e^{-d_2^2/2} \sqrt{T/(2\pi)}) D(V, \delta)
\]

and then the \( \gamma \)-cuts \( \text{dur}_\gamma = [\text{dur}_\gamma^-, \text{dur}_\gamma^+] \) can be expressed employing Proposition 1. In the two base cases of Section 3 and with \( T = 3 \) numerical computation yields \( \text{dur}_0 = [2.761, 3.076] \) for the A-rated bond and \( \text{dur}_0 = [2.137, 3.211] \) for the poorer-quality bond. The crisp values are 2.918 and 2.619, respectively. Note that the crisp value of the duration is smaller than \( T \), which is the duration of the equivalent default-free zero-coupon bond, but \( \text{dur}_\gamma^+ \) may be greater than \( T \). The results of numerical simulations for different maturities \( T \) is reported below in Figures 10 and 11. The fuzzy duration of the defaultable bond is delimited by the dashed line, while the solid line represents the duration of the equivalent default-free bond. We conclude that \( \text{dur}_\gamma < T \) is not always true, in keeping with some odd empirical outcomes.

### 5 Expectations under fuzzy goals

In this section we follow Yoshida (2003) and show how the valuation method works if fuzzy expectations are considered which takes the investor’s subjective utility function into account. Let \( \varphi \) denote a fuzzy goal, that is a continuous and increasing function from \([0, +\infty)\) to \([0, 1]\), such that \( \varphi(0) = 0 \) and \( \varphi(x) \to 1 \) as \( x \to +\infty \). The fuzzy expectation under \( \varphi \) is defined as in Yoshida (2003), (2.4) and a rational expected value is defined as the real number where the fuzzy expectation attains the supremum. The following argument is confined to determining the expected credit spread of a defaultable bond. A similar argument applies to the other related quantities.
Figure 10:

Figure 11:
Let the investor’s fuzzy goal, $\varphi$, be given. Then the investor’s permissible range of expected credit spreads under $\varphi$ is:

$$[x^{\hat{v}_-}, x^{\hat{v}_+}] = \left\{ x \in \mathbb{R}; \hat{s}_T^x(V, \delta)(x) \geq \varphi(x) \right\}.$$

It is the range of spreads $x$ such that the reliability degree of the optimal expected spread is greater than the degree of the investor’s satisfaction, $\varphi(x)$.

Going back to the base case we discussed in Section 3, we consider the following fuzzy goal:

$$\varphi(x) = 1 - e^{-0.01x} \text{ if } x \geq 0 \text{ and } 0 \text{ otherwise.}$$

The corresponding permissible range of expected credit spreads under this fuzzy goal, expressed in basis points, is $[48, 133]$, as Figure 12 shows. This set gives the investor’s confidence interval of expected credit spreads under randomness and fuzziness. Note that the crisp value, 95.38, is included in this range. Finally, by means of the FindRoot built-in-routine of Mathematica one can compute $\gamma^{\hat{v}_-}$ such that $\varphi(x^{\hat{v}_-}) = \hat{s}_T^{\hat{v}_-}(V, \delta)$. In our base case we obtain $\gamma^{\hat{v}_-} = 0.3824$ and $\gamma^{\hat{v}_+} = 0.7356$ that represent the degrees of investor’s satisfaction in the credit risk premium on the bond.

6 Conclusion

This paper shows that some difficulties with the structural modeling of credit risk can be partly removed throughout a fuzzy approach. One common criticism which is cast on the structural approach is that it results in less realistic predicted credit spreads than the reduced-form approach. However a reduced-form approach does not provide conceptual insights on credit risk. The advantage of combining a structural approach with a fuzzy framework is to model explicitly the mechanism behind default while introducing more flexibility than in traditional mod-
eling. We have shown that even a parsimonious model has the flexibility for capturing the basic features of the observed credit spread and duration of the bonds. This may open up further paths of research in credit risk analysis.

7 References

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